Restriction Categories as Double Categories

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Overview

Arrows in many categories have a flavour of *partiality*. Restriction categories are one of the attempts to axiomatise this phenomenon.

The goal of this talk is to introduce restriction categories and double categories and compare them. For the comparison, we choose the notion of product.

1 Restriction Categories

2 Double Categories

3 Products

Motivation – domains of definition via arrows

Set_p – category of sets and partial functions
Domain of definition $f : A \to B \mapsto dom(f) \in 2^A$
Can we describe this structure categorically? I.e. can we speak about arrows instead of subsets?

Definition

Let $f, g : A \to B$ be partial functions. We say that $f \leq g$ iff for every $a \in dom(f)$, we have $f(a) = g(a)$.

$2^A \cong \{f : A \to A : f \leq \mathrm{id}_A\} \rightsquigarrow$ Domain of definition $f : A \to B \mapsto \mathrm{id}_A|_{dom(f)}$
Composition corresponds to intersection of the domains of definition.
Note that if we call this assignment $\overline{(-)}$, then $f \leq g \iff f = g\overline{f}$

Restriction categories

Definition (Restriction category)

A restriction category consists of a category $\mathscr{A}$ together with for every $A, B \in \mathscr{A}$, a map $\overline{(-)} \colon \mathscr{A}(A,B) \to \mathscr{A}(A,A)$ such that

1. $f = f\overline{f}$
2. $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$
3. $\overline{g}\overline{f} = \overline{g\overline{f}}$
4. $f\overline{gf} = \overline{g}f$

(whenever these equations make sense).

Many consequences, e.g. $\overline{\mathrm{id}_A} = \mathrm{id}_A$, $\overline{f} \cdot \overline{f} = \overline{f}$, $\overline{\overline{f}} = \overline{f}$, ...

Examples

Example (Sets and partial functions)

Set_p: sets and partial functions

$$\overline{f} = \mathrm{id}_A|_{dom(f)}$$

Example (Trivial restrictions)

For every category $\mathscr{C}$, we can construct a restriction category $Triv(\mathscr{C})$: category remains the same and for $f : C \to D$, we set $\overline{f} := \mathrm{id}_C$.

Example (Partical recursive functions)

Single object $\mathbb{N}$ and partial recursive functions as morphisms

$$\overline{f} = \mathrm{id}_{\mathbb{N}}|_{dom(f)}$$

Examples

Example (Kleisli category for $-+1$)

Let $\mathscr{C}$ be a distributive category (i.e. finite products, finite coproducts and $A \times (B + C) \cong A \times B + A \times C$), then the Kleisli category for the $-+1$ monad can be equipped with the following restriction structure: For $f : A \to B + 1$, we set

$$\overline{f} := A \xrightarrow{\Delta} A \times A \xrightarrow{A \times f} A \times (B+1) \cong A \times B + A \xrightarrow{\pi_1 + !} A + 1.$$

Example (Stable lattices)

Let StabLat be the category, whose objects are meet-semilattices (i.e. posets with binary infima and top element) and morphisms are functions preserving binary infima (not necessarily the top element though!). Then $\mathrm{StabLat}^{op}$ has the following restriction structure: For $f \in \mathrm{StabLat}(L, K)$, we set

$$\overline{f}(k) := k \wedge f(\top_L) \colon K \to K.$$

Examples – categories of partial maps

Definition (Stable system of monics)

A stable system of monics on a category $\mathscr{C}$ is a collection $\mathcal{M}$ of monomorphisms that

- is closed under compositions;
- contains all isomorphisms;
- is pullback-stable (pullbacks along $m \in \mathcal{M}$ exist and are in $\mathcal{M}$ again).

Example

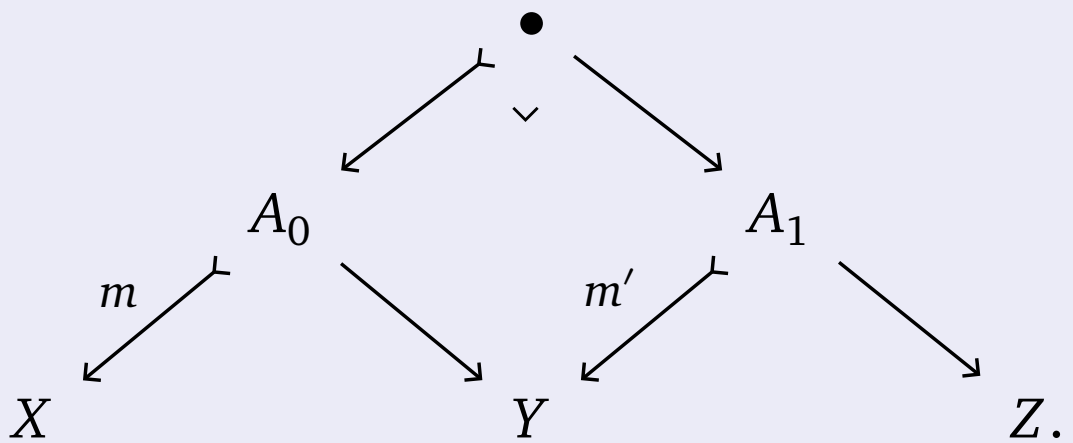
- All isomorphisms form a stable system of monics on every category.
- Monomorphisms in any category with pullbacks.
- Open subspaces of topological spaces.

Examples – Categories of partial maps

Definition (Category of partial maps)

Let $(\mathscr{C}, \mathcal{M})$ be a stable system of monics, we define a category $\mathrm{Par}_{\mathcal{M}}(\mathscr{C})$, whose

- objects are those of $\mathscr{C}$;
- morphisms $\mathrm{Par}_{\mathcal{M}}(\mathscr{C})(X, Y)$ are equivalence classes of spans $X \overset{m}{\leftarrowtail} A_0 \overset{f}{\to} Y$ with $m \in \mathcal{M}$;
- compositions:



Examples – Categories of partial maps

Example

The category of partial maps can be equipped with the structure of a restriction category with

$$\overline{X \overset{m}{\leftarrowtail} A_0 \overset{f}{\to} Y} := X \overset{m}{\leftarrowtail} A_0 \overset{m}{\rightarrowtail} X.$$

Total morphisms

Definition (Total morphism)

A morphism $f : A \to B$ is called total if $\overline{f} = \mathrm{id}_A$.

For partial (recursive) functions, totality is the expected notion, whereas for a trivial restriction structure, everything is total.
For stable lattices, total morphisms are those that actually preserve the top element and for $\mathrm{Par}_{\mathcal{M}}(\mathscr{C})$, we get spans of the form $X \overset{\mathrm{id}}{\leftarrow} X \to Y$.

Total morphisms of a restriction category $\mathscr{C}$ form a wide subcategory of $\mathscr{C}$ denoted $Total(\mathscr{C})$.

Order on hom sets

Definition (Order)

Let $f, g : A \to B$. We say that $f \leq g \iff f = g\overline{f}$.

In Set_p, this gives precisely what we introduced before.

Several nice properties:

- Every restriction category is enriched in posets.
- $\overline{f} \leq \mathrm{id}$
- If $f \leq g$ and f is total, then $f = g$.

Restriction functors

Definition (Restriction functor)

A functor $F : \mathscr{A} \to \mathscr{B}$ between restriction categories is called a restriction functor if for every f, $F(\overline{f}) = \overline{F(f)}$.

Example

- For every restriction category $\mathscr{A}$, the inclusion $Triv(Total(\mathscr{A})) \to \mathscr{A}$ is a restriction functor.
- The inclusion of partial recursive functions into Set_p is a restriction functor.

Proposition

Restriction functors preserve total morphisms and the order on hom sets.

1 Restriction Categories

2 Double Categories

3 Products

Definition of a double category

Many examples of categories with (at least) two types of arrows with a fruitful interaction. Double categories provide a way to formalize such contexts.

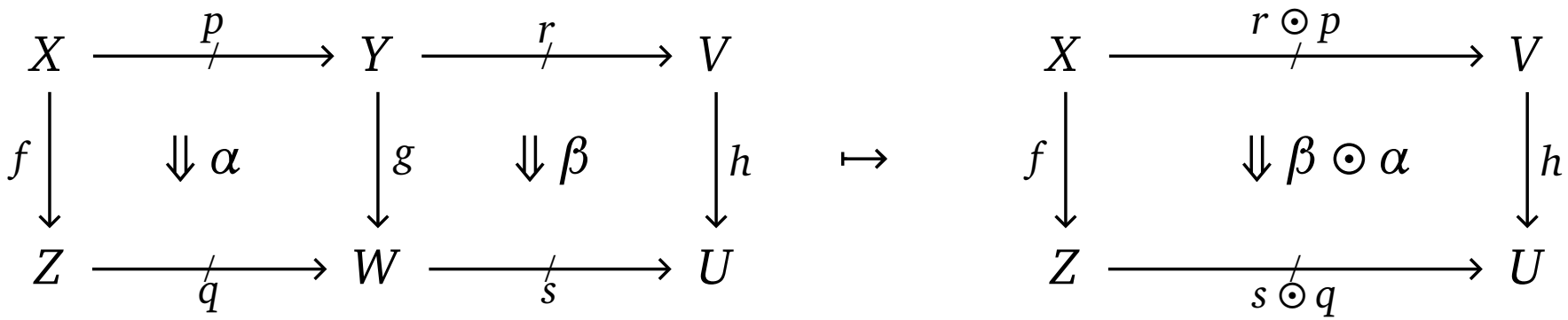
A *(strict) double category* $\mathbb{D}$ consists of

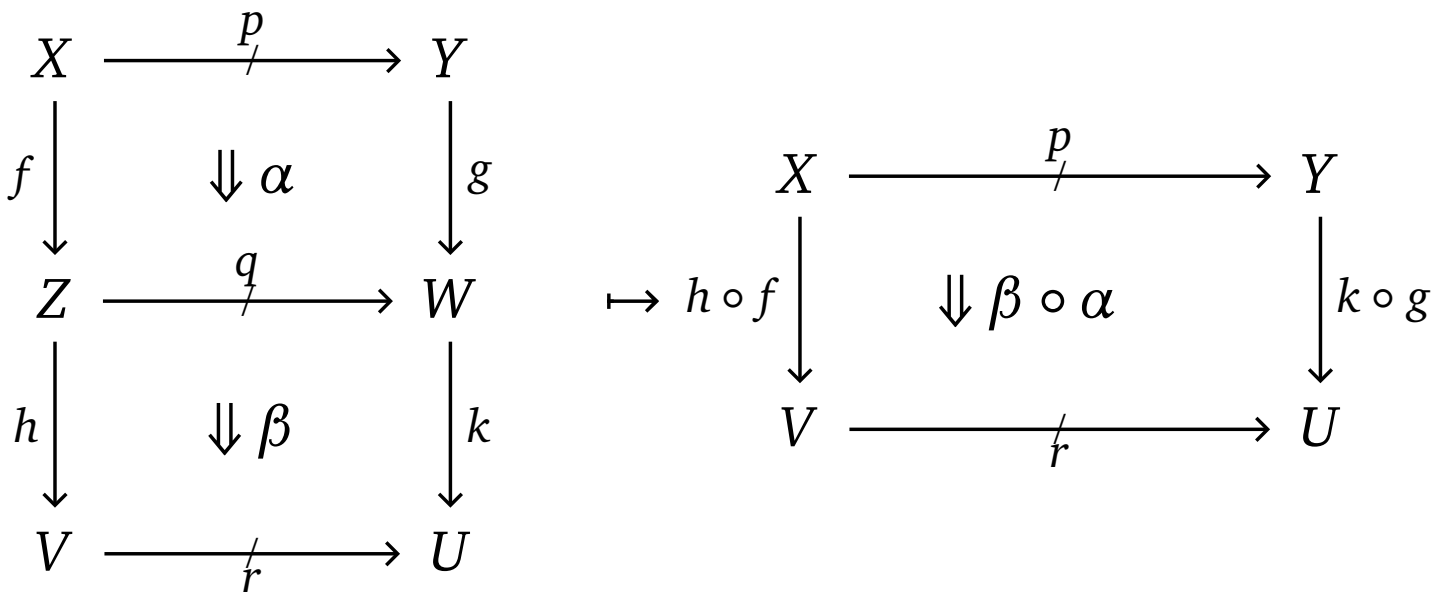
- objects $X, Y, Z, \ldots$
- *tight* arrows between objects $f : X \to Z$
- *loose* arrows between objects $p : X \nrightarrow Y$
- 2-cells of the form

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[q]{} & W \end{array}$$

Definition of a double category

- we can compose such squares both in the loose and the tight direction

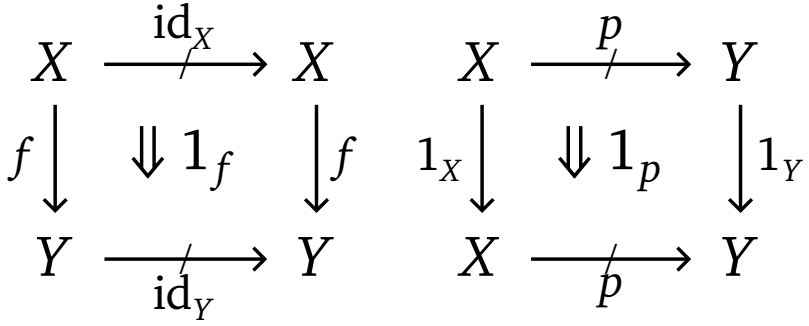




Both compositions are associative and compatible with each other: $(\delta \circ \gamma) \odot (\beta \circ \alpha) = (\delta \odot \beta) \circ (\gamma \odot \alpha)$.

Definition of a double category

- there are both tight $1_X : X \to X$ and loose $\mathrm{id}_X : X \nrightarrow X$ identities as well as 2-cell identities



with respect to the previous compositions.

Example (commutative squares)

From any category $\mathscr{C}$, we can form a double category in a trivial way by taking objects to be those of $\mathscr{C}$, loose and tight arrows to be the arrows of $\mathscr{C}$ and 2-cells to be commutative squares with the composition and identities inhereted from $\mathscr{C}$.

Examples of double categories

Example (relations)

The double category $\mathbb{R}\mathrm{el}(\mathrm{Set})$ of sets, functions and relations. The objects are sets. For any sets X, Y, a tight arrow $f : X \to Y$ is a function and a loose arrow $R : X \nrightarrow Y$ is a relation, i.e. $R \subseteq X \times Y$. For any sets X, Y, Z, W, functions f, g and relations R, S there is a unique 2-cell

$$\begin{array}{ccc} X & \xrightarrow{R} & Y \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[S]{} & W \end{array}$$

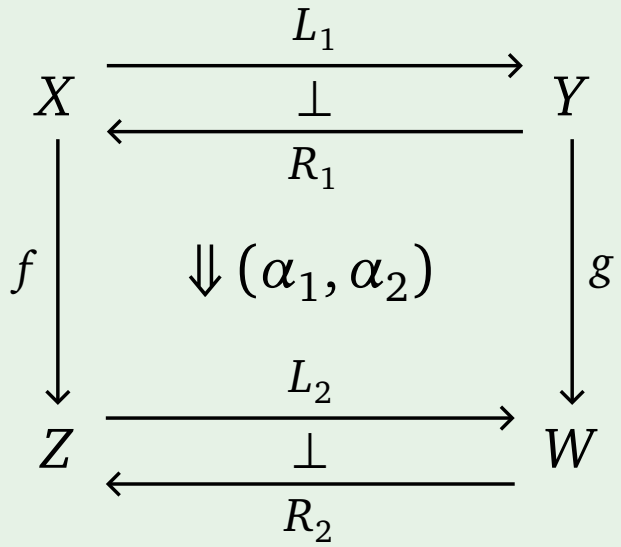
exactly when $x \sim_R y \implies f(x) \sim_S g(y)$. Tight composition is function composition while loose composition is composition of relations. Identities are the evident ones.

Remark. More generally, we can consider morphisms and relations in any *regular* category.

Examples of double categories

Example (adjunctions)

From any 2-category $\mathscr{C}$ we can form the double category $\mathbb{A}\mathrm{dj}(\mathscr{C})$ of adjunctions in $\mathscr{C}$. The objects are those of $\mathscr{C}$, the tight arrows are the arrows of $\mathscr{C}$ and the loose arrows are adjunctions internal to $\mathscr{C}$. A 2-cell



is a pair of mate 2-cells in $\mathscr{C}$, i.e. $\alpha_1 : gL_1 \Rightarrow L_2 f$ and $\alpha_2 : fR_1 \Rightarrow R_2 g$. Both compositions and identities are inherited from $\mathscr{C}$. Composition of adjunctions is an adjunction!

Restriction categories as double categories

From any restriction category $\mathscr{A}$ we can get a double category $\mathbb{R}(\mathscr{A})$.

- objects: objects of $\mathscr{A}$
- tight arrows: *total* maps, i.e. arrows in $\mathscr{A}$ for which $\overline{f} = 1$
- loose arrows: (all) arrows of $\mathscr{A}$
- 2-cells: For any $X, Y, Z, W \in \mathscr{A}_0$, f, g total maps and u, v arrows in $\mathscr{A}$, a unique 2-cell

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[v]{} & W \end{array}$$

  exists if and only if $gu \leqslant vf \iff gu = vf\,\overline{gu} = vf\,\overline{u}$.
- Both compositions and identities: those inhereted from $\mathscr{A}$.

Examples of double restriction categories

Example ($\mathbb{R}(\mathrm{Set}_p)$)

For the restriction category Set_p of sets and partial functions, the double category $\mathbb{R}(\mathrm{Set}_p)$ has sets as objects, total functions as tight arrows and partial (not necessarily total) functions as loose arrows. A 2-cell exists if and only if the two composites of the square agree on where the top is defined.

Example ($\mathbb{R}(\mathrm{Par}_{\mathcal{M}}(\mathscr{C}))$)

Suppose $\mathscr{C}$ is a category with a stable system of monics $\mathcal{M}$ and form the category of partial arrows $\mathrm{Par}_{\mathcal{M}}(\mathscr{C})$ with respect to $\mathcal{M}$. The induced double category has objects those of $\mathscr{C}$, loose arrows are spans of the form

$$X \overset{m}{\leftarrowtail} A_0 \overset{f}{\to} Y,$$

and tight arrows spans for which $m = 1_X$, i.e. plain arrows in $\mathscr{C}$.

Examples of double restriction categories

Example ($\mathbb{R}(\mathsf{StabLat}^{\mathrm{op}})$)

The double category $\mathbb{R}(\mathsf{StabLat}^{\mathrm{op}})$ has meet-semilattices as objects, stable maps as loose arrows and stable maps that preserve $\top$ as tight arrows. A 2-cell

$$\begin{array}{ccc} X & \xrightarrow{\ u\ } & Y \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[\ v\ ]{} & W \end{array}$$

exists if and only if

$$gu = vf\overline{u} \iff gu(x) = vf\left(x \wedge u(\top_X)\right) \quad \forall x \in X$$

Double functors

A **(strict) double functor** $F : \mathbb{C} \to \mathbb{D}$ between (strict) double categories $\mathbb{C}, \mathbb{D}$, sends:

- objects to objects,
- loose arrows to loose arrows,
- tight arrows to tight arrows,
- 2-cells

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[v]{} & W \end{array}$$

to 2-cells

$$\begin{array}{ccc} FX & \xrightarrow{Fu} & FY \\ {\scriptstyle Ff}\downarrow & \Downarrow F\alpha & \downarrow{\scriptstyle Fg} \\ FZ & \xrightarrow[Fv]{} & FW \end{array}$$

respecting compositions and identities.

Double restriction functors

For restriction categories $\mathscr{A}, \mathscr{B}$, restriction functors $F : \mathscr{A} \to \mathscr{B}$ induce double functors

$$\mathbb{R}F : \mathbb{R}(\mathscr{A}) \to \mathbb{R}(\mathscr{B})$$

defined as follows:

- on objects: $\mathbb{R}F(X) := F(X)$,
- on loose arrows: $\mathbb{R}F(f) := Ff$;
- on tight (total) arrows: $\mathbb{R}F(f) := Ff$;
  ($\mathbb{R}F$ preserves total arrows since F does)
- on 2-cells:

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[v]{} & W \end{array} \quad \mapsto \quad \begin{array}{ccc} FX & \xrightarrow{Fu} & FY \\ {\scriptstyle Ff}\downarrow & \Downarrow F\alpha & \downarrow{\scriptstyle Fg} \\ FZ & \xrightarrow[Fv]{} & FW \end{array}$$

($\mathbb{R}F$ sends the unique 2-cell to the unique 2-cell since F preserves ordering)

Posed questions

Restriction functors $F : \mathscr{A} \to \mathscr{B}$ induce double functors $\mathbb{R}F : \mathbb{R}(\mathscr{A}) \to \mathbb{R}(\mathscr{B})$.

Question: Does the converse hold?

Not in general. Every such $\mathbb{R}F$ is uniquely determined by a *functor* $\mathscr{A} \to \mathscr{B}$ which

- preserves total morphisms
- preserves the partial order on homsets
- $\overline{Ff} \leqslant F\overline{f}$.

We have sufficient conditions, e.g.

Proposition

Every double functor $\mathbb{R}(\mathrm{Par}_{\mathcal{M}}(\mathscr{C})) \to \mathbb{R}(\mathrm{Par}_{\mathcal{N}}(\mathscr{D}))$ comes from a restriction functor $\mathrm{Par}_{\mathcal{M}}(\mathscr{C}) \to \mathrm{Par}_{\mathcal{N}}(\mathscr{D})$.

But not a complete characterization.
Task: Find necessary and sufficient conditions.

1 Restriction Categories

2 Double Categories

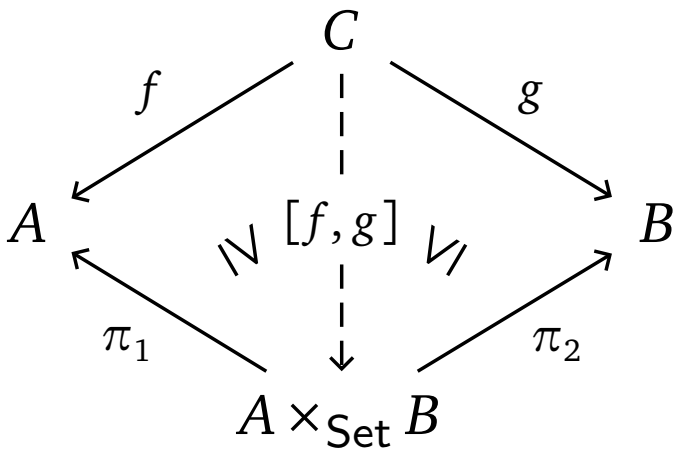
3 Products

Products in Set_p

Set_p has products, but they are quite unexpected!

$$A \times_{\mathsf{Set}_p} B \cong A + A \times_{\mathsf{Set}} B + B$$

What are interesting properties of $A \times_{\mathsf{Set}} B$ in Set_p then?
Given $f : C \to A, g : C \to B$, we can still define $[f,g] : C \to A \times_{\mathsf{Set}} B$ by $[f,g](c) = (f(c), g(c))$ if the right-hand side is defined and undefined otherwise.

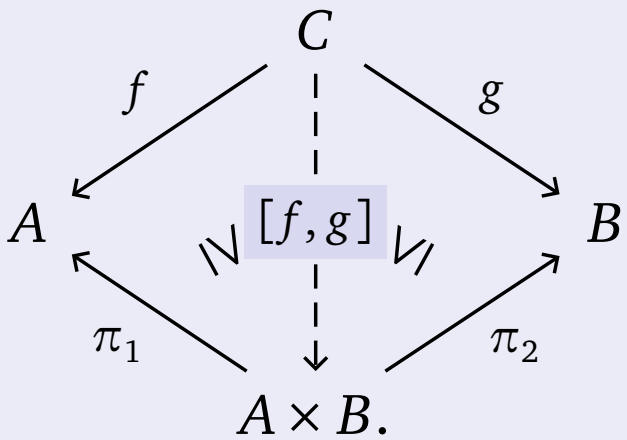


$[f,g]$ is the unique arrow satisfying the above inequalities and $dom([f,g]) = dom(f) \cap dom(g)$

Restriction products

Definition (Restriction products)

Let $\mathscr{A}$ be a restriction category and $A, B \in \mathscr{A}$. Then their restriction product consists of an object $A \times B$ together with two total projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that for every $f : C \to A, g : C \to B$ there exists a unique arrow $[f, g] : C \to A \times B$ such that $\overline{[f, g]} = \overline{f}\,\overline{g}$ and



Restriction products give actual products in $Total(\mathscr{A})$.

Restriction products – examples

Example (Set_p)

Restriction products in Set_p are precisely products in Set.

More generally:

Example (Kleisli Category for $-+1$)

Let $\mathscr{C}$ be a distributive category. Then the Kleisli (restriction) category Kl_{+1} has restriction products and these are precisely products in $\mathscr{C}$.

Double products

Definition (Double products)

A double category $\mathbb{D}$ has binary double products if for every $A, B \in \mathbb{D}$, there is an object $A \times B$ together with tight projections $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ such that

- $A \times B, \pi_1, \pi_2$ form a product cone in the tight category of $\mathbb{D}$;
- for every loose arrows $h : A \nrightarrow C, h' : B \nrightarrow D$, we have a loose arrow $h \times h' : A \times B \nrightarrow C \times D$ together with two squares

$$\begin{array}{ccc} A \times B & \xrightarrow{h \times h'} & C \times D \\ {\scriptstyle \pi_1}\downarrow & \Downarrow \Pi_1 & \downarrow{\scriptstyle \pi_1} \\ A & \xrightarrow[h]{} & C \end{array} \qquad \begin{array}{ccc} A \times B & \xrightarrow{h \times h'} & C \times D \\ {\scriptstyle \pi_2}\downarrow & \Downarrow \Pi_2 & \downarrow{\scriptstyle \pi_2} \\ B & \xrightarrow[h']{} & D \end{array}$$

such that $h \times h'$ with Π_1, Π_2 is a product cone in the category of loose arrows and squares.

Double products of relations

Since the existence of double products implies the existence of products in the tight category, we do not get anything surprising on objects.

Example ($\mathbb{R}$el(Set))

The product of objects A, B is $A \times B$.
The product of relations $R \subseteq A \times C$ and $S \subseteq B \times D$ is the relation $R \times S \subseteq (A \times B) \times (C \times D)$ defined by

$$(a, b) \sim_{R \times S} (c, d) \iff a \sim_R c \text{ and } b \sim_S d.$$

The first horizontal projection Π_1 is the implication

$$(a, b) \sim_{R \times S} (c, d) \implies a \sim_R c.$$

Comparison

Conjecture

A restriction category $\mathscr{A}$ has binary restriction products iff its associated double category $\mathbb{R}(\mathscr{A})$ has binary double products.

We can similarly define restriction terminal objects and double terminal objects and conjecture this correspondence also for terminal objects.

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